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**List of the results in the paper**

# THE COMBINATORIAL SUM $\sum_{k\equiv r \pmod m}\binom{n}{k}$ AND ITS APPLICATIONS IN NUMBER THEORY III

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**Notations.**

$\mathbb{Z}$—the set of integers, $\mathbb{N}$—the set of positive integers.

For $a, b \in \mathbb{Z}$ the Lucas sequence $\{u_n(a,b)\}$ is defined by

$$u_0(a,b)=0,\ u_1(a,b)=1 \text{ and } u_{n+1}(a,b)=bu_n(a,b)-au_{n-1}(a,b)\ (n\ge 1).$$

So $F_n = u_n(-1,1)$ is the Fibonacci sequence, and $u_n = u_n(-1,2)$ is just the Pell sequence.

In the paper $[x]$ denotes the integral part of $x$, $\{x\}(=x-[x])$ denotes the fractional part of $x$, $(\frac{a}{p})$ denotes the Legendre symbol, and $q_p(a)=(a^{p-1}-1)/p$.

## 3.1 Introduction and basic lemmas.

**Lemma 3.1.** *Let $p, m, k, s$ be positive integers. Then*

$$\sum_{\substack{a=0\\ a\equiv sp \pmod m}}^{p-1} a^k \equiv (-m)^k \sum_{\frac{(s-1)p}{m}<a\le \frac{sp}{m}} a^k \pmod p.$$

**Corollary 3.1.** *Let $p$ be an odd prime, $m, s \in \mathbb{N}$ and $p \nmid m$. Then*

$$m\sum_{\substack{k=1\\ k\equiv sp \pmod m}}^{p-1} \frac{1}{k} \equiv -\sum_{\frac{(s-1)p}{m}<k<\frac{sp}{m}} \frac{1}{k} \pmod p.$$

**Corollary 3.2.** *Let $p$ be an odd prime, $m \in \mathbb{N}$ and $p \nmid m$. Then*

$$\sum_{k=1}^{[\frac{2p}{m}]} \frac{(-1)^k}{k} \equiv m\sum_{\substack{k=1\\ k\equiv 2p \pmod m}}^{p-1} \frac{1}{k} \pmod p.$$

**Corollary 3.3.** *Let $p$ be an odd prime, $k,m,s \in \mathbb{N}$, $2 \nmid m$ and $p \nmid m$. Then*

$$\sum_{\frac{(s-1)p}{m}<k<\frac{sp}{m}} \frac{(-1)^k}{k} \equiv m(-1)^s \sum_{\substack{k=1 \\ k\equiv sp (\mathrm{mod}\ m)}}^{p-1} \frac{(-1)^{k-1}}{k} \pmod p.$$

**3.2 $F_{p-(\frac{5}{p})}/p$ and $u_{p-(\frac{2}{p})}/p$.**

For $m,p \in \mathbb{N}$ and $s \in \mathbb{Z}$ let

$$K_m(s,p) = \sum_{\substack{k=1 \\ k\equiv sp (\mathrm{mod}\ m)}}^{p-1} \frac{1}{k}.$$

**Lemma 3.2.** *Let $p$ be an odd prime, $m \in \mathbb{N}$, $s,t \in \mathbb{Z}$, $p \nmid m$ and $s+t \equiv 1 \pmod m$. Then*

$$K_m(s,p) \equiv -K_m(t,p) \pmod p.$$

**Theorem 3.1.** *Let $p \neq 2,5$ be a prime, $q_p(a) = (a^{p-1}-1)/p$, and let $F_n$ be the Fibonacci sequence. Then*

(i) $10K_{10}(1,p) \equiv -10K_{10}(0,p) \equiv 2q_p(2) + \frac{5}{4}q_p(5) + \frac{15}{4} \cdot \frac{F_{p-(\frac{5}{p})}}{p} \pmod p$,

(ii) $10K_{10}(2,p) \equiv -10K_{10}(9,p) \equiv -2q_p(2) - \frac{5}{2} \cdot \frac{F_{p-(\frac{5}{p})}}{p} \pmod p$,

(iii) $10K_{10}(3,p) \equiv -10K_{10}(8,p) \equiv 2q_p(2) - 5\frac{F_{p-(\frac{5}{p})}}{p} \pmod p$,

(iv) $10K_{10}(4,p) \equiv -10K_{10}(7,p) \equiv -2q_p(2) + \frac{5}{2} \cdot \frac{F_{p-(\frac{5}{p})}}{p} \pmod p$,

(i) $10K_{10}(5,p) \equiv -10K_{10}(6,p) \equiv 2q_p(2) - \frac{5}{4}q_p(5) + \frac{5}{4} \cdot \frac{F_{p-(\frac{5}{p})}}{p} \pmod p$.

**Theorem 3.2.** *Let $p \neq 2,5$ be a prime. Then*

(1) (Z.H.Sun and Z.W.Sun,1992) $\frac{F_{p-(\frac{5}{p})}}{p} \equiv -2 \sum_{\substack{k=1 \\ k\equiv 2p (\mathrm{mod}\ 5)}}^{p-1} \frac{1}{k} \pmod p$;

(2) (H.C.Williams, 1991) $\frac{F_{p-(\frac{5}{p})}}{p} \equiv \frac{2}{5} \sum_{\frac{p}{5}<k<\frac{2p}{5}} \frac{1}{k} \pmod p$;

(3) $\frac{F_{p-(\frac{5}{p})}}{p} \equiv \frac{2}{5} \sum_{1\le k<\frac{2p}{5}} \frac{(-1)^{k-1}}{k} \pmod p$;

(4) (H.C.Williams, 1982) $\frac{F_{p-(\frac{5}{p})}}{p} \equiv -\frac{2}{5} \sum_{1\le k<\frac{4p}{5}} \frac{(-1)^{k-1}}{k} \pmod p$;

(5) $\frac{F_{p-(\frac{5}{p})}}{p} \equiv \frac{2}{5} \sum_{\frac{p}{5}<k<\frac{p}{3}} \frac{(-1)^{k}}{k} \pmod p \ (p \neq 3)$;

(6) $\frac{F_{p-(\frac{5}{p})}}{p} \equiv 6\Big( \sum_{\substack{k=1 \\ k\equiv 4p (\mathrm{mod}\ 15)}}^{p-1} \frac{(-1)^{k-1}}{k} - \sum_{\substack{k=1 \\ k\equiv 5p (\mathrm{mod}\ 15)}}^{p-1} \frac{(-1)^{k-1}}{k}\Big) \pmod p \ (p \neq 3)$;

(7) $\frac{F_{p-(\frac{5}{p})}}{p} \equiv -\frac{4}{3} \sum_{\substack{k=1 \\ k\equiv 2p,3p \pmod{10}}}^{p-1} \frac{1}{k} \equiv \frac{2}{15} \sum_{\frac{p}{10}<k<\frac{3p}{10}} \frac{1}{k} \pmod p \ (p \neq 3);$

(8) $\frac{F_{p-(\frac{5}{p})}}{p} \equiv \frac{4}{5}\big((-1)^{[p/5]}\binom{p-1}{[p/5]} - 1\big)/p - q_p(5) \pmod p.$

We remark that Theorem 3.2(5) and (6) provide a quick way to calculate the Fibonacci quotient $F_{p-(\frac{5}{p})}/p$.

**Theorem 3.3.** *Let $p$ be an odd prime, $u_n = u_n(-1,2)$, and $q_p(2) = (2^{p-1}-1)/p$. Then*

(i) $8K_8(1,p) \equiv -8K_8(0,p) \equiv 4q_p(2) + 2u_{p-(\frac{2}{p})}/p \pmod p$;

(ii) $8K_8(2,p) \equiv -8K_8(7,p) \equiv -q_p(2) - 2u_{p-(\frac{2}{p})}/p \pmod p$;

(iii) $8K_8(3,p) \equiv -8K_8(6,p) \equiv q_p(2) - 2u_{p-(\frac{2}{p})}/p \pmod p$;

(iii) $8K_8(4,p) \equiv -8K_8(5,p) \equiv -2q_p(2) + 2u_{p-(\frac{2}{p})}/p \pmod p$.

**Theorem 3.4.** *Let $p$ be an odd prime and $u_n = u_n(-1,2)$. Then*

(i) $\frac{u_{p-(\frac{2}{p})}}{p} \equiv -2q_p(2) + \frac{1}{2}\big((-1)^{[p/8]}\binom{p-1}{[p/8]} - 1\big)/p \pmod p$.

(ii) $\frac{u_{p-(\frac{2}{p})}}{p} \equiv -2 \sum_{\substack{k=1 \\ k\equiv 2p,3p \pmod 8}}^{p-1} \frac{1}{k} \equiv \frac{1}{4} \sum_{\frac{p}{8}<k<\frac{3p}{8}} \frac{1}{k} \pmod p;$

(iii) (Z.W.Sun) $\frac{u_{p-(\frac{2}{p})}}{p} \equiv \frac{1}{2} \sum_{\frac{p}{4}<k<\frac{p}{2}} \frac{(-1)^k}{k} \pmod p;$

(iv) $\frac{u_{p-(\frac{2}{p})}}{p} \equiv \frac{1}{4}\Big( \sum_{\frac{p}{8}<k<\frac{p}{4}} \frac{1}{k} + \sum_{\frac{p}{4}<k<\frac{p}{3}} \frac{(-1)^k}{k}\Big) \pmod p \ (p \neq 3).$

In 1991, H.C.Williams proved that

$$\frac{u_{p-(\frac{2}{p})}}{p} \equiv \frac{1}{4} \sum_{\frac{p}{8}<k<\frac{3p}{8}} \frac{1}{k} \pmod p.$$

**3.3** $F(p)/p \pmod p$**.**

Let $E_n(x)$ be the Euler polynomials given by

$$E_n(x) + \sum_{r=0}^{n} \binom{n}{r} E_r(x) = 2x^n \quad (n = 0,1,2,\dots).$$

**Lemma 3.3.** *Let $p$ be an odd prime, $x, y, n \in \mathbb{Z}$, $x \equiv y \pmod p$ and $0 \le n \le p-2$. Then*

$$E_n(x) \equiv E_n(y) \pmod p.$$

**Theorem 3.5.** *Let $F(0) = 1,\ F(1) = 0,\ F(2) = 2$ and $F(n+2) = 3F(n) - F(n-1)$ $(n = 1,2,3,\dots)$. If $p > 3$ is a prime, then $p \mid F(p)$ and*

$$\frac{F(p)}{p} \equiv (-1)^{[\frac{5p}{9}]}\frac{1}{6}E_{p-2}(\{\frac{5p}{9}\}) - \frac{1}{3}q_p(2) \pmod p.$$

### 3.4 $5^{\frac{p-1}{4}} \pmod{p}$.

**Proposition 3.1.** *Let $p$ be a prime of the form $4k+1$, and $p = a^2 + b^2$ with $2 \nmid a$ and $2 \mid b$.*

(i) (Gauss) *If $p \equiv 1 \pmod 8$, then $4 \mid b$ and*

$$2^{\frac{p-1}{4}} \equiv (-1)^{\frac{b}{4}} \pmod{p};$$

(ii) (Dirichlet) *If $p \equiv 5 \pmod 8$, $a \equiv 1 \pmod 4$ and $b \equiv 2 \pmod 8$, then*

$$2^{\frac{p-1}{4}} \equiv b/a \pmod{p}.$$

**Lemma 3.4.** *Let $p$ be an odd prime, $a, b \in \mathbb{Z}$, $p \nmid a(b^2 - 4a)$, $(\frac{a}{p}) = 1$ and $c^2 \equiv a \pmod{p}$.*

(i) *If $(\frac{b^2-4a}{p}) = 1$, then*

$$u_{\frac{p+1}{2}}(a, b) \equiv \Big(\frac{b-2c}{p}\Big) \pmod{p}, \quad u_{\frac{p-1}{2}}(a, b) \equiv 0 \pmod{p}.$$

(ii) *If $(\frac{b^2-4a}{p}) = -1$, then*

$$u_{\frac{p+1}{2}}(a, b) \equiv 0 \pmod{p}, \quad u_{\frac{p-1}{2}}(a, b) \equiv \frac{1}{c}\Big(\frac{b-2c}{p}\Big) \pmod{p}.$$

**Theorem 3.6.** *Let $p > 5$ be a prime of the form $4k+1$, and $p = a^2 + b^2$ with $a, b \in \mathbb{Z}$ and $2 \mid b$.*

(i) (Gauss) *If $p \equiv 1, 9 \pmod{20}$, then*

$$5^{\frac{p-1}{4}} \equiv \begin{cases} 1 \pmod{p} & \text{if } 5 \mid b, \\ -1 \pmod{p} & \text{if } 5 \mid a. \end{cases}$$

(ii) *If $p \equiv 13, 17 \pmod{20}$ and $a \equiv b \pmod 5$, then*

$$5^{\frac{p-1}{4}} \equiv b/a \pmod{p}.$$

**Corollary 3.4.** (E.Lehmer, 1966) *Let $p \equiv 1, 9 \pmod{20}$ be a prime, and $p = a^2 + b^2$ with $a, b \in \mathbb{Z}$ and $2 \mid b$.*

(i) *If $p \equiv 1, 29 \pmod{40}$, then $p \mid F_{\frac{p-1}{4}} \iff 5 \mid b$;*

(ii) *If $p \equiv 9, 21 \pmod{40}$, then $p \mid F_{\frac{p-1}{4}} \iff 5 \mid a$.*

**Conjecture 3.1.** *Let $p \equiv 5 \pmod{12}$ be a prime, and $p = a^2 + b^2$ with $b \equiv 0 \pmod 2$ and $a \equiv -b \pmod 3$. Then*

$$(-3)^{\frac{p-1}{4}} \equiv b/a \pmod{p}.$$

**Conjecture 3.2.** *Let $p \equiv 3 \pmod 8$ be a prime, and $p = x^2 + 2y^2$ with $x \equiv 5, 7 \pmod 8$ and $y \equiv 3 \pmod 4$. Then*

$$u_{\frac{p+1}{4}}(-1, 2) \equiv \frac{1}{2}(-1)^{[\frac{x+y}{4}]} \pmod{p}.$$